Binomial Distribution

<u>Assumptions</u>

1. The experiment consists of n of Bernoulli trials.
2. The probability of a success is the same for each trial.
3. The trials are independent.
4. X denote the no. of successes obtained in the n trials

Binomial Distribution

Definition

A random variable X is said to have binomial distribution with parameter n and p if its density is given

$$f(x) = b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n.$$

where p is the probability of getting success.

Binomial Distribution

Example: An experiment consists of four tosses of a coin. Denoting the outcomes *HHTH*, *THTT*, ….., and assuming that all 16 outcomes are equally likely, find the probability distribution for the total number of head.

Solution: $n = 4$ and $p = 1/2$, Let X be the total number of heads in 4 tosses where $x = 0, 1, 2, 3, 4,$ the probability distribution for X is given by

$$f(x) = P(X = x) = b(x{:}4,1/2)$$

Binomial Distribution

Distribution function for binomial distribution

$$F(x) = \sum_{k=0}^{x} b(k; n, p) \text{ for } x = 0, 1, 2, \ldots, n$$

$$= \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} \text{ for } x = 0, 1, 2, \ldots, n$$

Binomial Distribution

- The value of $b(x;n,p)$ can be obtained by formula

$$b(x;n,p) = B(x;n,p) - B(x-1;n,p)$$

Note: If n is large the calculation of binomial probability can become quite tedious.

Binomial Distribution Table

$$F(x) = B(x; n, p) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

Table for $n = 5$ and $p = .1$ to $.4$

		p				
n	x	0.1	0.2	0.25	0.3	0.4
5	0	0.5905	0.3277	0.2373	0.1681	0.0778
	1	0.9185	0.7373	0.6328	0.5282	0.3370
	2	0.9914	0.9421	0.8965	0.8369	0.6826
	3	0.9995	0.9933	0.9844	0.9692	0.9130
	4	1	0.9997	0.9990	0.9976	0.9898
	5	1	1	1	1	1

Calculation of probabilities

$$a.\quad B(8;16,0.40) = 0.8577$$

$$b.\quad b(8;16,0.40) = B(8;16,0.4) - B(7;16,0.4)$$
$$= 0.8577 - 0.7161 = 0.1416$$

$$c.\quad \sum_{k=6}^{20} b(k;20,0.15) = 1 - B(5;20,0.15) = 1 - 0.9327$$
$$= 0.0673$$

$$d.\quad \sum_{k=2}^{4} b(k;9,0.30) = B(4;9,0.30) - B(1;9,0.30)$$
$$= 0.9012 - 0.1960 = 0.7052$$

Mean, Variance and Moment generating function of Binomial distribution

Mean

$$\mu = n \cdot p$$

$p \rightarrow$ probability of success
$n \rightarrow$ number of trials

Variance

$$\sigma^2 = n.p.(1 - p)$$

Moment generating function

$$M_X(t) = \left(q + pe^t\right)^n.$$

Binomial Distribution

Proof for Mean $\boxed{\mu = n \cdot p}$

$p \rightarrow$ probability of success
$n \rightarrow$ number of trials

$$\mu = \sum_{x=0}^{n} x.b(x;n,p) = \sum_{x=0}^{n} x.\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x.\frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Binomial Distribution

$$\mu = np\sum_{x=1}^{n}\frac{(n-1)!}{(x-1)!(n-x)!}p^{x-1}(1-p)^{n-x}$$

Put $x - 1 = y$, so $n - x = n - 1 - y$,

$$\mu = np\sum_{y=0}^{n-1}\frac{n-1!}{y!(n-1-y)!}p^{y}(1-p)^{n-1-y}$$

$$= np(p+1-p)^{n-1}$$

$$= np$$

Binomial Distribution

Proof forVariance: $\sigma^2 = n.p.(1-p)$

$$\mu_2' = \sum_{x=0}^{n} x^2.b(x;n,p) = \sum_{x=0}^{n} x^2.\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x^2.\frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Binomial Distribution

$$\mu_2' = np\sum_{x=1}^{n} x.\frac{(n-1)!}{(x-1)!(n-x)!}p^{x-1}(1-p)^{n-x}$$

Put $x - 1 = y$ and $n - 1 = m$

$$\mu_2' = np\sum_{y=0}^{m} (y+1).\frac{m!}{y!(m-y)!}p^{y}(1-p)^{m-y}$$

Binomial Distribution

$$\mu_2' = np\sum_{y=0}^{m} y.\frac{m!}{y!(m-y)!}p^y(1-p)^{m-y}$$

$$+np\sum_{y=0}^{m}\frac{m!}{y!(m-y)!}p^y(1-p)^{m-y}$$

$$= np^2 m\sum_{y=1}^{m}\frac{(m-1)!}{(y-1)!(m-y)!}p^{y-1}(1-p)^{m-y}$$

$$+np$$

Binomial Distribution

Put $y - 1 = z$ and $m - 1 = l$ in first summation

$$\mu_2' = np^2(n-1)\sum_{z=0}^{l}\frac{l!}{z!(l-z)!}p^z(1-p)^{l-z} + np$$

$$= np^2(n-1) + np = n^2p^2 - np^2 + np$$

$$\sigma^2 = \mu_2' - \mu^2 = np(1-p)$$

Binomial Distribution

Proof for Moment genrating function

$$m.g.f = E\left(e^{tX}\right) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, 2, ..., n$$

$$= \sum_{x=0}^{n} \binom{n}{x} \left(pe^t\right)^x q^{n-x}$$

$$= \left(q + pe^t\right)^n$$

m.g.f (continue)

$$Mean = \left.\frac{dM}{dt}\right|_{t=0} = n\left(q + pe^t\right)^{n-1} pe^t\Big|_{t=0} = n\left(q + p\right)^{n-1} p$$

$$= np$$

Second ordinary/raw moment (moment about origin)

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \left[n(n-1)p\left(q + pe^t\right)^{n-2}\left(pe^t\right)(e^t) + np\left(q + pe^t\right)^{n-1}\left(e^t\right)\right]\Big|_{t=0}$$

$$= n(n-1)p^2 + np$$

$$\boxed{\sigma^2 = \mu_2' - \mu^2}$$

$$\sigma^2 = n(n-1)p^2 + np - (np)^2$$

$$= (np)^2 - np^2 + np - (np)^2$$

$$= np(1-p) = npq$$

Example-2

An agricultural cooperative claims that 90% of the watermelons shipped out are ripe and ready to eat. Find the probability that among 18 watermelons shipped out

(a) All 18 are ripe and ready to eat;

(b) At least 16 are ripe and ready to eat;

(c) At most 14 are ripe and ready to eat.

Solution

$$\text{(b) } P(X \geq 16) = \sum_{k=16}^{18} b(k;18,0.90) = 1 - B(15;18,0.90)$$

$$= 1 - 0.2662 = 0.7338$$

$$\text{(c) } P(X \leq 14) = B(14;18,0.90) = 0.0982$$

(a): Given $n = 18$, $p = 0.90$

$P(X = 18) = \text{b}(18;18,0.90) = 1 - B(17;18,0.90)$

$= 0.1501$

Example-3

Suppose that 90% of all copies of a particular textbook fails a certain binding strength test. If 15 copies selected at random,

(a) find the probability that at most 8 copies pass the test.

(b) $P(2<X<6)$ if X is the no of copies which pass the test.

Example-4

It is known that balls produced by a certain company will be defective with Probability 0.01 and independently of each other. The company sells the balls in package of 10 and offers a money back guarantee that at most 1 of the 10 balls is defective. What proportion of the packages sold must the company replaced ?

Ans: 0.004

Example-5

It is known that CDs produced by a certain company will be defective with probability 0.01 and independent of each other. The company sells the CDs in a packages of size 10 and offers a money back guarantee that at most one of the 10 CDs in the package will be defective. If some one buy 3 packages, what is the probability that he or she will return exactly one of them ?